LOCAL RIGIDITY OF COMPLEX HYPERBOLIC LATTICES IN SEMISIMPLE LIE GROUPS

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ABSTRACT. We show the local rigidity of complex hyperbolic lattices in classical Hermitian semisimple Lie groups, $SU(p,q), Sp(2n+2,\mathbb{R}), SO^*(2n+2), SO(2n,2)$. This reproves or generalizes some results in [2, 9, 11, 15].

1. INTRODUCTION

After the seminal work of A. Weil [19, 17], many pioneering works about local rigidity of lattices in semisimple Lie groups have been done by Raghunathan, Matsushima-Murakami, Goldman-Millson and others. As in quasi-Fuchsian deformation of Fuchsian groups, one could expect a possible deformation of a lattice of a semisimple Lie group L inside a larger Lie group $G \supset L$. Due to Margulis superrigidity of higher rank semisimple Lie groups, and to Corlette's superrigidity of $Sp(n,1), F_{4(-20)}$, there is no local deformation for lattices in such semisimple Lie groups. Hence a natural interesting problem is to study lattices of $L = SO(n,1)$ and $SU(n,1)$ in $G \supset L$. A local rigidity of lattices of $L = SO(3,1)$ and $SO(4,1)$ inside $G = Sp(n,1)$ is proved in [8]. A local rigidity of complex hyperbolic uniform lattices Γ of $SU(n,1)$ inside $SU(n+1,1)$ is first studied by Goldman-Millson [2]. Note that in this case $H^1(\Gamma, \mathfrak{su}(n+1,1))$ decomposes as $H^1(\Gamma,\mathbb{R}) \oplus H^1(\Gamma,\mathbb{C}^{n+1})$. What Goldman-Millson showed is that there is no deformation coming from $H^1(\Gamma,\mathbb{C}^{n+1})$. Indeed there is a deformation coming from $H^1(\Gamma,\mathbb{R})$. But this deformation corresponds to the less interesting deformation obtained by deforming Γ in $U(n,1)$ by a curve of homomorphism into the centralizer $U(1)$ of $SU(n,1)$ in $U(n,1)$. Henceforth, when we say 'locally rigid', we ignore this kind of deformation through the centralizer. A further generalization of this case to quaternionic Kähler manifolds is studied in [9, 11]. In those papers the embeddings of $L = SU(n,1)$ are in the classical Lie groups and are obtained by the standard (or natural) embeddings $\rho : L \to G$ of L in $G = Sp(n,1), SU(2n,2), SO(4n,4)$.

The infinitesimal rigidity result above is partly determined by the cohomology group $H^1(\Gamma,\mathfrak{g})$. It is known by Raghunathan [16] that the cohomology group $H^1(\Gamma, W)$ of Γ acting on an irreducible representation space W of $L = SU(n,1)$ vanishes unless W is a symmetric tensor power $S(\mathbb{C}^{n+1})$ (or its dual) of the standard representation $\mathbb{C}^{n+1}$. In a recent paper [11] Klingler proved two results on local rigidity for uniform lattices of L in

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G via $\rho : L \to G$. To recall his results we denote $M = S(U(n) \times U(1)) \subset L$ and $K \subset G$ the maximal compact subgroups of the respective semisimple Lie groups, and $\mathfrak{m}, \mathfrak{l}, \mathfrak{k}, \mathfrak{g}$ the corresponding Lie algebras. Let $Z_K(L)$ be the centralizer of L in $K \subset G$ and $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ its Lie algebra. The first result [11] states that if there exists a non-zero element $Z \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ acting on the representation spaces $S^m(V)$ with certain positivity property (see (4) below for a precise statement) then there is local rigidity. The second result weakens the assumption by replacing $Z \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ by the condition $Z \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{m})$, namely by Z being in a (generally) larger subspace $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{m}) \supset \mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$. The proof of both results uses a cohomology theory of polarized real variation of Hodge structures.

The purpose of the present paper is two-fold. We shall prove first a general local rigidity result by assuming existence of certain elements $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ by using elementary computations for the usual cohomology of differential forms instead of polarized real variation of Hodge structures. Secondly we consider some natural homomorphisms $\rho : L = SU(n,1) \to G$ into the classical groups $SU(p,q), Sp(2n+2,\mathbb{R}), SO^*(2n+2), SO(2n,2)$. We explain briefly how the homomorphisms $\rho : L \to G$ are constructed. The first embedding into $SU(p,q)$ is through the diagonal embedding of L into $SU(nq_0, q_0)$. There are involutions on the groups $G = Sp(2n+2,\mathbb{R}), SO^*(2n+2), SO(2n,2)$ whose fixed point subgroup is precisely $U(n,1)$, namely $U(n,1)$ is a symmetric subgroup of G and $G/U(n,1)$ is a non-Riemannian symmetric space. We examine further the decomposition of $\mathfrak{g}$ under L. In case that the symmetric tensor representations $S^m(\mathbb{C}^{n+1})$ of L do not appear in the decomposition the rigidity follows immediately from the vanishing theorem of Raghunathan [16]. If on the contrary there are such summands appearing, then we use Theorem 1.1 below to prove the local rigidity.

Throughout this paper, $\Gamma \subset SU(n,1),\ n \geq 2$ is a uniform lattice. We list the homomorphisms just mentioned:

(i) The diagonal homomorphism of $L = SU(n,1)$ in $SU(p,q)$,
(ii) Satake homomorphism in $G = Sp(n+1,\mathbb{R})$, and
(iii) Ihara homomorphisms into Hermitian Lie groups $SO^*(2n)$ and $SO(2n,2)$.

Our main results are the following. The precise notations will be defined in the next section. Let $\rho : L = SU(n,1) \to G$ be a Lie group injective homomorphism. For simplicity we shall view L as a subgroup of G. Recall that $Z_K(L) = Z_K(\rho(L))$ the centralizer of $\rho(L)$ in K and $Z_G(L)$ the centralizer in G. Let $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ and $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{l})$ be the respective Lie algebras. Let $\mathfrak{g}$ be the Lie algebra of G and $B(\cdot,\cdot)$ the Killing form. The space $\mathfrak{g}$ is then a real representation of L, $\mathrm{Ad} \circ \rho : L \mapsto \mathrm{End}(\mathfrak{g})$, via ρ and the adjoint representation of G on $\mathfrak{g}$. As L is real semisimple the space $\mathfrak{g}$ is decomposed into irreducible representations. Denote $V = \mathbb{C}^{n+1}$, and write its decomposition under $U(n)$ as $V = V_1 + \mathbb{C}$, with $V_1 = \mathbb{C}^n$, the standard representation of $U(n)$. The symmetric powers $S^m(V)$ and $S^m(\mathbb{C}^n)$, $m \geq 1$ are irreducible representations of L and $U(n)$, respectively. For simplicity we will denote the representation spaces by S^m and s^m. We have

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_0 \tag{1}$$

where

$$\mathfrak{g}_1 = \sum_m S^m \otimes \mathbb{R}^{d_m} \tag{2}$$

is the sum of isotypes of S^m, and $\mathfrak{g}_0$ is of different isotype from S^m. In particular each space $S^m \otimes \mathbb{R}^{d_m}$ is a representation of $\mathfrak{l} + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l})$. Consider the subspace $s^m \subset S^m$ and its M-isotypes in $S^m \otimes \mathbb{R}^{d_m}$,

$$W_m := s^m \otimes \mathbb{R}^{d_m} \subset S^m \otimes \mathbb{R}^{d_m}. \tag{3}$$

The complex multiplication by i on the real space W_m will be written simply as usual by $X \to iX$, keeping in mind that all linear forms involved are real linear forms. It can also be written as the Lie algebra action of a center element T_0 of $\mathfrak{u}(n)$; see §2 below.

Theorem 1.1. *Let $L = SU(n,1),\ n \geq 2$ and $\Gamma \subset L$ a uniform lattice. Suppose there exist an L-invariant bilinear form b on $\mathfrak{g}^{\mathbb{C}}$ and an element $Z \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ such that*

$$-b(\mathrm{ad}(Z)(X), iX) = -b(Z, [X, iX]) > 0, \forall X \in W_m, m \geq 1. \tag{4}$$

Then $\rho : \Gamma \subset L \to G$ is locally rigid.

Theorem 1.2. *In the totally geodesic embeddings (i), (ii), (iii) above, a uniform lattice $\Gamma \subset SU(n,1) \subset G$ is locally rigid.*

The claim for the Satake and Ihara homomorphisms answers partly a question of Pansu posed to us. Recently Pozzetti [15, Corollary 1.5] proved also a local rigidity theorem for the diagonal homomorphism in (1) using the geometry of Shilov boundary of bounded symmetric domains. See also [12] for a different technique.

We shall present an elementary and independent proof of the cases in Theorem 1.2.

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2. Preliminaries

We fix first some notation and convention. If $W = \mathbb{C}^n$ is a complex vector space we shall denote $W_{\mathbb{R}} = \mathbb{R}^{2n}$ the underlying real vector space by forgetting the complex structure. In particular if W is a complex representation space of a real reductive Lie group L (acting as complex linear transformation) the above notation $W_{\mathbb{R}}$ makes sense as a real representation of L.

A standard representation of a $n \times n$ real or complex matrix group G refers to a standard matrix group action on the vector space $\mathbb{R}^n$ or $\mathbb{C}^n$. The complexification of a real Lie algebra $\mathfrak{g}$ will be denoted by $\mathfrak{g}^{\mathbb{C}}$. The Killing form on $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$ will be denoted by $B(\cdot,\cdot)$.

Let $V = \mathbb{C}^{n,1}$ be the space $\mathbb{C}^{n,1}$ equipped with the Hermitian form $(x,y)_J = (Jx, y)$ of signature $(n,1)$, where J is the diagonal matrix $J = \mathrm{diag}(1, \cdots, 1, -1)$ and (x,y) the standard Hermitian form. The group $L = SU(n,1)$ consists of complex linear transformations on V preserving the form $(x,y)_J$ and of determinant 1. The symmetric space L/M, $M = S(U(n) \times U(1)) = U(n) \subset L$ being the maximal compact subgroup, can be realized

as the unit ball B in $\mathbb{C}^n$, $B = L/M$ with $z = 0$ being the base point. In $(n+1)\times(n+1)$-matrix realization of $g \in SU(n,1)$ the action is given as follows

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in L, z \in B \mapsto w = gz = (az+b)(cz+d)^{-1}. \tag{5}$$

The Lie algebra $\mathfrak{u}(n,1)$ consists of matrices X such that $X^*J + JX = 0$ and $\mathfrak{l} = \mathfrak{su}(n,1)$ the subspaces of elements with trace 0. Let $\mathfrak{l} = \mathfrak{u}(n) + \mathfrak{q}$ be the Cartan decomposition of $\mathfrak{l}$. Here $\mathfrak{u}(n) = s(\mathfrak{u}(n) + \mathfrak{u}(1))$ consists of traceless block-diagonal skew-Hermitian matrices. The tangent space $T_{x_0}(B)$ at x_0 will be identified with $\mathfrak{q} = \mathbb{C}^n$ as a real space.

We fix a central element T_0 of the maximal compact subalgebra $\mathfrak{u}(n)$

$$T_0 = (n+1)^{-1}\sqrt{-1}\text{diag}(1, \cdots, 1, -n).$$

Let $V_1 = \mathbb{C}^n$ be the standard representation of $U(n)$ and $\det$ be the determinant representation. The element T_0 defines the complex structure on B and we have

$$\mathfrak{sl}(n+1,\mathbb{C}) = \mathfrak{sl}(n,\mathbb{C}) + \mathbb{C}T_0 + \mathfrak{q}^+ + \mathfrak{q}^- \tag{6}$$

where $\mathfrak{q}^\pm$ are the holomorphic and anti-holomorphic tangent spaces at $z = 0 \in B$. As a representation of $\mathfrak{u}(n)$ we have $\mathfrak{q}^+ = V_1 \otimes \det$.

The standard representation V of $\mathfrak{l}$ under $\mathfrak{u}(n)$ is

$$V = V_1 \oplus \det^{-1}.$$

Note that we have

$$S^j(V) = \oplus_{k=0}^j S^k(V_1) \otimes \det^{k-j}, \quad S^j(V') = \oplus_{k=0}^j S^k(V_1') \otimes \det^{j-k}, \tag{7}$$

where $S^k(V_1)$ is the symmetric tensor of the standard representation V_1 of $\mathfrak{su}(n)$ and V' is the dual representation of V.

3. General rigidity results

3.1. **Discrete groups and automorphic cohomologies.** Let Γ be a cocompact lattice of $L = SU(n,1),\ n \geq 2$ and ρ a representation of Γ in a complex vector space W. The infinitesimal deformations of ρ are described by the cohomology group $H^1(\Gamma, W)$. We refer to [17] for an account of the underlying theory. We have the following result of Raghunathan [16].

Theorem 3.1 (Raghunathan). *Let $\rho : \mathbf{SU}(n,1) \longrightarrow \mathbf{GL}(W)$ be a real finite dimensional irreducible representation of $\mathbf{SU}(n,1), n \geq 2$. Let Γ be a cocompact lattice in $SU(n,1)$. Then $H^1(\Gamma, W) = 0$ except if $W \simeq S^jV$ for some $j \geq 0$, where S^j denotes the j-th symmetric power.*

Given a uniform lattice $\Gamma \subset L$ and a holomorphic representation s of the complexification $M^{\mathbb{C}} = GL(n,\mathbb{C})$ of M there is also an automorphic holomorphic bundle over $X = \Gamma\backslash L/M$ and the corresponding automorphic cohomology $H^{p,q}(X, \Gamma, s)$. In [13, 14] general relations between the discrete group cohomology associated to a representation of G and the automorphic cohomology associated to a holomorphic representation of $M^{\mathbb{C}}$ are studied. Refined

relations between them have also been further obtained for general Hermitian Lie groups; see e.g. [20]. In [9, 10, 11] it is proved for $L = SU(n, 1)$ that the cohomology $H^1(\Gamma, S^j)$ is isomorphic to an automorphic $(0, 1)$-cohomology taking values in s^j. We formulate it in terms of the tangent and the canonical bundle. Let $T = T^{(1,0)}$ be the holomorphic tangent bundle of X and $\mathcal{L}^{-1}$ be the line bundle on X defined so that $\mathcal{L}^{-(n+1)}$ is the canonical line bundle K_X. Recall the notation $V = \mathbb{C}^{n+1}$ and $V_1 = \mathbb{C}^n$, the representation spaces of L and M.

Theorem 3.2. *Any element $\alpha \in H^1(\Gamma, S^j(V))$ can be realized as a $S^j(\mathbb{C}^n)$-valued $(0, 1)$-form. This realization induces an isomorphism*

$$H^1(\Gamma, S^j(V)) = H^1(X, E(S^j(V_1))) = H^{(0,1)}(X, S^jTX \otimes \mathcal{L}^{-j}).$$

For a complete proof, see Theorem 1.4.1 of [11] or Theorem 1.1 of [10].

Remark 3.3. *The cohomology group $H^1(\Gamma, S^j(V))$ can be represented by harmonic forms with values in a holomorphic vector bundle over L/M and there is further a decomposition $H^1(\Gamma, S^j(V)) = H^{(0,1)}(\Gamma, S^j(V)) + H^{(1,0)}(\Gamma, S^j(V))$. Generally there is an injection of the first part $H^{(0,1)}(\Gamma, S^j(V))$ into the automorphic cohomology $H^1(X, E(S^j(V_1)))$; see* [13]*. The above theorem implies that the injection is also surjective and that second term vanishes. In the present paper we shall use only the injective property of $H^1(\Gamma, S^j(V))$ into the automorphic cohomology.*

3.2. **Proof of Theorem 1.1.**

Proof. We use similar arguments as in [9] for quaternionic forms combined with the Eichler-Shimura isomorphism for the representation $S^m(V)$ of L. Assume that $\phi \in H^1(\Gamma, \mathfrak{g})$ is a non-zero element. Recall that we have a bilinear form $[\cdot, \cdot] : H^1(\Gamma, \mathfrak{g}) \times H^1(\Gamma, \mathfrak{g}) \to H^2(\Gamma, \mathfrak{g})$, $[\alpha \otimes X, \beta \otimes Y,](u, v) = (\alpha(u)\beta(v) - \alpha(v)\beta(u))[X, Y]$. If ϕ represents a non-trivial deformation then we have $[\phi, \phi] = 0$ as an element of $H^2(\Gamma, \mathfrak{g})$, that is $[\phi, \phi] = d\eta$ for some one co-chain η. We shall prove that this leads to a contradiction. We will freely view ϕ and η as differential forms on G; see [17]. Recall the decomposition (1) and the subspaces $\mathfrak{g}_1$ and W_m defined in (2)- (3). It follows first by Theorem 3.1 that ϕ takes values in $\mathfrak{g}_1$. Theorem 3.2 asserts further that any element in $H^1(\Gamma, S^m(\mathbb{C}^{n+1}))$ is a $(0, 1)$-form taking value in the subspace $s^m = S^m(\mathbb{C}^n)$. See also [10] for an elementary proof. Thus $\phi = \sum_m \phi_m$ with ϕ_m being W_m-valued $(0, 1)$-forms.

Now let $Z \in \mathfrak{z}_\mathfrak{k}(\mathfrak{l})$ be as in the assumption. In particular Z is L-invariant. It defines then a non-zero element in the cohomology $H^0(\Gamma, \mathfrak{g})$, the subspace of Γ invariant elements in $\mathfrak{g}$ via the action $\mathrm{Ad} \circ \rho$. We consider the bilinear form

$$\xi(X, Y) = b(Z, [\phi(X), \phi(Y)]) = b([Z, \phi(X)], \phi(Y)).$$

The Lie bracket $[X, Y]$ defines a M-equivariant map $(\sum_m W_m) \otimes (\sum_m W_m) \to \mathfrak{g}$ and we have

$$[\phi, \phi] = \sum_{m,m'} [\phi_m, \phi_{m'}]$$

and

$$b(Z,[\phi(X),\phi(Y)])=\sum_{m,m'}b(Z,[\phi_m(X),\phi_{m'}(Y)])$$

However for any $m\neq m'$ the spaces W_m and $W_{m'}$ are inequivalent representations of M, b and Z are L-invariant, thus it follows from Schur's lemma [5, 6.1] that

$$b(Z,[\phi_m(X),\phi_{m'}(Y)])=0$$

for any X, Y. Indeed, the bilinear form $b(Z,[\phi_m,\phi_{m'}])$ defines also an M-equivariant map from $W_{m'}$ to $(W_m)^*=W_m$, and it must be zero by Schur's lemma.

Consequently

$$b(Z,[\phi(X),\phi(Y)])=\sum_{m}b(Z,[\phi_m(X),\phi_m(Y)]).$$

Using the fact that the ϕ_m are (0,1)-forms and our assumption, when we plug in an orthonormal basis of the underlying real Euclidean vector space of $\mathbb{C}^n$, $(E_1, iE_1\cdots, E_n, iE_n)$, there exists some $c_x>0$ on each $x\in\Gamma\backslash L/M$ such that

$$\begin{aligned}\frac{\xi\wedge\omega^{n-1}}{\omega^n}=\sum_{k=1}^n\xi(E_k,iE_k)&=\sum_{k=1}^n\sum_m b(Z,[\phi_m(E_k),\phi_m(iE_k)])\\&=-\sum_{k=1}^n\sum_m b(Z,[\phi_m(E_k),i\phi_m(E_k)])\geq c_x|\phi|_x^2.\end{aligned}$$

Here ω is a Kähler form on $H^n_{\mathbb{C}}=L/M$.

Put $\xi=b(Z,[\phi,\phi])=\lambda\circ[\phi,\phi]$ where λ is a linear functional. Therefore

$$\lambda\circ[\phi,\phi]\wedge\omega^{n-1}\geq c|\phi|^2\omega^n.$$

Here $c>0$ since Γ is a uniform lattice. On the other hand $[\phi,\phi]=d\eta$ and hence

$$\int_{\Gamma\backslash L/M}\lambda\circ[\phi,\phi]\wedge\omega^{n-1}=0,$$

a contradiction. □

3.3. **Proof of Theorems 1.2.** This follows from Theorem 3.1, Theorem 1.1 and Lemmas 4.1-4.3.

4. Embeddings of the unit ball in classical symmetric spaces

In this section we construct natural homomorphisms of $L=SU(n,1)$ in classical Lie groups $G=SU(p,q), Sp(p,\mathbb{R}), SO^*(2p), SO(2,p)$, and we prove the relevant decomposition of the Lie algebra $\mathfrak{g}$ under $\mathfrak{su}(n,1)$. We examine the condition in the statement of Theorem 1.1. The Killing form for these classical Lie algebra will be fixed once for all as $B(X,Y)=\mathrm{tr}XY$. In the $SO^*(2p)$ and $SO(2,p)$ cases, we will show that $S^m(V)$ factor does not appear in the Lie algebra decomposition, hence by Theorem 3.1, the local rigidity follows. (Certain decompositions of the complex Lie algebras here can also be done as in [1].)

4.1. **The diagonal homomorphism of $L = SU(n,1)$ in $G = SU(p,q)$.** We consider first the case $G = SU(nq_0, q_0)$. Let $W := V \otimes \mathbb{C}^{q_0} = \mathbb{C}^{n,1} \otimes \mathbb{C}^{q_0}$ be equipped with the Hermitian inner product $(\cdot,\cdot)_W := (\cdot,\cdot)_J \otimes (\cdot,\cdot)$, where $(\cdot,\cdot)$ is the standard Hermitian inner product on $\mathbb{C}^{q_0}$. The groups $L = SU(n,1)$ and $U(n,1)$ are diagonally embedded in $SU(W) = SU(nq_0, q_0)$ and respectively in $U(W) = U(nq_0, q_0)$ via $g \to g \otimes I_{\mathbb{C}^{q_0}}$, and we fix this realization. To find L-invariant forms as in Theorem 1.1 it is natural to consider the centralizer of L in $SU(nq_0, q_0)$. Indeed the group $U(q_0)$ is also diagonally embedded in G via $h \to I_{\mathbb{C}^{n,1}} \otimes h$, and commutes with $U(n,1)$ in $SU(nq_0, q_0)$. (The pair $(U(n,1), U(q_0))$ is an example of the so-called Howe dual pairs [4].) The space of L-invariant bilinear forms on $\mathfrak{su}(nq_0, q_0)$ forms then a representation space of $U(q_0)$ and it is therefore conceptually clear to consider the decomposition of $\mathfrak{su}(nq_0, q_0)$ first under $L \times U(q_0)$.

Consider the Lie algebra $\mathfrak{u}(nq_0, q_0)$ of $U(W) = U(nq_0, q_0)$. Let $\mathcal{H} = \mathcal{H}(q_0) = \{Y \in M_{q_0,q_0}; Y^* = Y\}$ be the space of $q_0 \times q_0$-Hermitian matrices Y viewed as a representation space of $U(q_0)$, namely $h \in U(q_0) : Y \mapsto hYh^*$; $\mathcal{H}$ is identified with the Lie algebra $\mathfrak{u}(q_o) = i\mathcal{H}$ of $U(q_0)$ as a representation space. It is immediate that the Lie algebra $\mathfrak{u}(nq_0, q_0)$ under the standard action of $U(n,1) \times U(q_0)$ is decomposed as

$$\mathfrak{u}(nq_0, q_0) = \mathfrak{u}(n,1) \otimes \mathcal{H}.$$

Indeed any element of the form $X \otimes Y \in \mathfrak{u}(n,1) \otimes \mathcal{H}$ is clearly an element in $\mathfrak{u}(nq_0, q_0)$, since

$$\begin{aligned}(X \otimes Y(v_1 \otimes u_1), v_2 \otimes u_2)_W &= (Xv_1, v_2)_J (Yu_1, u_2) = -(v_1, Xv_2)_J (u_1, Yu_2) \\ &= -(v_1 \otimes u_1, X \otimes Y(v_2 \otimes u_2))_W,\end{aligned}$$

and the whole space is generated by elements of this form by counting the dimensions.

We have now $\mathfrak{u}(n,1) = \mathfrak{l} + i\mathbb{R}I_{C^{n,1}}$, $\mathcal{H} = \mathbb{R}I_{\mathbb{C}^{q_0}} + \mathcal{H}_0$ where $\mathcal{H}_0$ is the subspace of trace free elements. By taking the trace free part and observing that $\mathrm{tr}(X \otimes Y) = (\mathrm{tr}X)(\mathrm{tr}Y)$ we find

$$\mathfrak{su}(nq_0, q_0) = \mathfrak{l} \otimes \mathcal{H} + iI_{C^{n,1}} \otimes \mathcal{H}_0.$$

To see how the Lie algebra $\mathfrak{l}$ is realized as a subalgebra we write the above formula as

$$\begin{aligned}\mathfrak{su}(nq_0, q_0) &= (\mathfrak{l} \otimes I_{\mathbb{C}^{q_0}} + iI_{\mathbb{C}^{n,1}} \otimes \mathcal{H}_0) + \mathfrak{l} \otimes \mathcal{H}_0 \\ &= (\mathfrak{l} + \mathfrak{su}(q_0)) + \mathfrak{l} \otimes \mathcal{H}_0\end{aligned} \tag{8}$$

under the action of $L \times U(q_0)$. Here we have used the above identification of L and $U(q_0)$ as well as their Lie algebras, so that $\mathfrak{l} = \mathfrak{l} \otimes I_{\mathbb{C}^{q_0}}$, $\mathfrak{su}(q_0) = iI_{\mathbb{C}^{n,1}} \otimes \mathcal{H}_0$.

We consider further the homomorphism of $SU(n,1)$ in $G = SU(p,q)$ via the above embedding $SU(n,1) \to SU(nq_0, q_0)$ and the natural inclusion $SU(nq_0, q_0) \subset SU(p,q)$ for $p \geq nq_0, q \geq q_0$ and $p + q > (n+1)q_0$. More precisely let $p_1 = p - nq_0, q_1 = q - q_0$ and let $\mathbb{C}^{p,q} = \mathbb{C}^{nq_0,q_0} \oplus \mathbb{C}^{p_1,q_1}$ be equipped with the indefinite Hermitian form defined by the matrix

$$\mathrm{diag}(J_1, J_2), \text{ where } J_1 = \mathrm{diag}(I_{nq_0}, -I_{q_0}),\ J_2 = \mathrm{diag}(I_{p_1}, -I_{q_1}).$$

The group $SU(p,q)$ is then $SU(\mathbb{C}^{p,q})$. The Lie algebra elements in $\mathfrak{g}$ will be written as 2×2-block matrices under the above decomposition of $\mathbb{C}^{p,q}$, and they are of the form

$$\begin{bmatrix} a & b \\ -J_2b^*J_1 & d \end{bmatrix},$$

with $a\in\mathfrak{u}(nq_0,q_0), d\in\mathfrak{u}(p_1,q_1)$, tr$a+trd=0$. Let

$$E:=\begin{bmatrix} i(p_1+q_1)I_{nq_0+q_0} & 0 \\ 0 & -i(n+1)q_0I_{p_1+q_1} \end{bmatrix}\in\mathfrak{g},$$

be the sum of two central elements in $\mathfrak{u}(nq_0,q_o)$ and respectively in $\mathfrak{u}(p_1,q_1)$, which is further in $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ according to our notation in Section 1. Thus we have

$$\mathfrak{g}=\mathfrak{su}(nq_0,q_0)\oplus\mathfrak{su}(p_1,q_1)\oplus\mathbb{R}E\oplus(\mathbb{C}^{nq_0,q_0}\otimes_{\mathbb{C}}(\mathbb{C}^{p_1,q_1})')_{\mathbb{R}} \tag{9}$$

as a representation space of $\mathfrak{su}(nq_0,q_0)+\mathfrak{su}(p_1,q_1)$ with $\mathbb{C}^{nq_0,q_0}$ the standard action of $\mathfrak{su}(nq_0,q_0)$, $(\mathbb{C}^{p_1,q_1})'$ the dual action of $\mathfrak{su}(p_1,q_1)$ and $\mathbb{R}E$ the trivial representation of $\mathfrak{su}(nq_0,q_0)+\mathfrak{su}(p_1,q_1)$.

The last factor $\mathbb{C}^{nq_0,q_0}\otimes_{\mathbb{C}}(\mathbb{C}^{p_1,q_1})'$ can be seen as follows. The matrices $A\in U(nq_0,q_0)$ and $B\in U(p_1,q_1)$ act on

$$\begin{bmatrix} 0 & b \\ -J_2b^*J_1 & 0 \end{bmatrix},$$

by

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} 0 & b \\ -J_2b^*J_1 & 0 \end{bmatrix}\begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}=\begin{bmatrix} 0 & AbB^{-1} \\ * & 0 \end{bmatrix}.$$

Now the subgroup $L\times SU(q_0)\subset SU(nq_0,q_0)\subset G$ acts on $\mathbb{C}^{nq_0,q_0}$ as $\mathbb{C}^{n,1}\otimes\mathbb{C}^{q_0}$ and thus on the last summand in (9) as

$$\mathbb{C}^{nq_0,q_0}\otimes(\mathbb{C}^{p_1,q_1})'=\mathbb{C}^{n,1}\otimes\mathbb{C}^{q_0}\otimes(\mathbb{C}^{p_1,q_1})'.$$

The first summand is treated in (8), and the formula (9) now reads

$$\begin{aligned}\mathfrak{g}=(\mathfrak{l}+\mathfrak{su}(q_0))\oplus(\mathfrak{l}\otimes\mathcal{H}_0)\oplus\mathfrak{su}(p_1,q_1)\oplus\mathbb{R}E\\ \oplus(\mathbb{C}^{n,1}\otimes_{\mathbb{C}}\mathbb{C}^{q_0}\otimes_{\mathbb{C}}(\mathbb{C}^{p_1,q_1})')_{\mathbb{R}}.\end{aligned} \tag{10}$$

Note here that the diagonal embedding of $SU(n,1)$ in G is via $L\times SU(q_0)\subset SU(nq_0,q_0)\subset G$ and hence it commutes with $SU(p_1,q_1)$. Therefore the above decomposition is under the diagonal embedding of $SU(n,1)$. Therefore the standard representation $\mathbb{C}^{n,1}$ of L appears with multiplicity $q_0(p_1+q_1)$. Using the notation in Section 1 we have

$$W_1=\mathbb{C}^n\otimes\mathbb{C}^{q_0}\otimes(\mathbb{C}^{p_1,q_1})',$$

consisting of matrices

$$X(b)=\begin{bmatrix} 0 & b \\ -J_2b^*J_1 & 0 \end{bmatrix},$$

with

$$b=\begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix},$$

b_{11} of size $nq_0\times p_1$ and b_{12} of size $nq_0\times q_1$.

Assume first $p_1 = p - nq_0 > 0$. We let

$$Z = \mathrm{diag}(i\alpha I_{(n+1)q_0}, i\beta I_{p_1}, i\gamma I_{q_1}) \in \mathfrak{z}\mathfrak{k}(\mathfrak{l}),$$

where α, β, γ are real numbers subjected to $(n+1)q_0\alpha + p_1\beta + q_1\gamma = 0$ so that tr$Z = 0$, and will be chosen afterwards. In the computation below we shall suppress the index m of I_m.

We compute the action of Z on X and find

$$[Z, X(b)] = \begin{bmatrix} 0 & v \\ -J_2 v^* J_1 & 0 \end{bmatrix},$$

with

$$v = i\alpha I \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix} - i \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \end{bmatrix} \mathrm{diag}(\beta I, \gamma I) = \begin{bmatrix} i(\alpha-\beta)b_{11} & i(\alpha-\gamma)b_{12} \\ 0 & 0 \end{bmatrix}.$$

We check now the condition in Theorem 1.1. Replacing b by ib we have

$$X(ib) = \begin{bmatrix} 0 & ib \\ iJ_2 b^* J_1 & 0 \end{bmatrix}.$$

The bilinear form $B([Z, X(b)], X(ib))$ in question is

$$B([Z, X(b)], X(ib)) = \mathrm{tr}[Z, X(b)]X(ib) = -2(\alpha-\beta)\mathrm{tr} b_{11}b_{11}^* - 2(\gamma-\alpha)\mathrm{tr} b_{12}b_{12}^*$$

by a direct computation. We choose now $\beta = -1$, γ any real number such that

$$\frac{p_1}{(n+1)q_0 + q_1} < \gamma < \frac{(n+1)q_0 + p_1}{q_1}$$

(which clearly exists) and

$$\alpha = \frac{p_1 - \gamma q_1}{(n+1)q_0}$$

Then we have indeed $(n+1)q_0\alpha + p_1\beta + q_1\gamma = 0$, and

$$\gamma > \alpha > \beta$$

so that $B([Z, X(b)], X(ib))$ is negative definite.

If $p_1 = p - nq_0 = 0$ then $q_1 = q - q_0 > 0$. We replace Z above by the following matrix

$$Z = \mathrm{diag}(-i\frac{q_1}{(n+1)q_0} I_{(n+1)q_0}, iI_{q_1})$$

The same computation as above shows that we still have

$$B([Z, X], iX) = -2\mathrm{tr} b^* b < 0$$

whenever $b \neq 0$.

Summarizing we have

Lemma 4.1. *Consider the diagonal embedding of* $L = SU(n, 1)$ *in* $G = SU(p, q)$ *via* $L \times U(q_0) \subset G$ *for* $p \geq nq_0, q \geq q_0$

(1) *Suppose* $p = nq_0, q = q_0$. *The Lie algebra* $\mathfrak{g}$ *is decomposed under* $\mathfrak{su}(n,1) + \mathfrak{su}(q)$ *as*

$$\mathfrak{g} = \mathfrak{su}(nq_0, q_0) = (\mathfrak{su}(n,1) \otimes \mathcal{H}(q)) \oplus (I_{\mathbb{C}^{n,1}} \otimes \mathfrak{su}(q)) = (\mathfrak{l} + \mathfrak{su}(q)) \oplus (\mathfrak{l} \otimes \mathcal{H}_0(q))$$

where $\mathcal{H}(q)$ *is the space of* $q \times q$*-Hermitian matrices viewed as a representation space of* $\mathfrak{su}(q)$ *and* $\mathcal{H}_0(q)$ *is the trace free part. Thus no symmetric tensor* $S^m V$ *of* $\mathfrak{su}(n,1)$ *appears in the decomposition under the diagonal embedding of* $SU(n,1)$ *in* G.

(2) *Suppose* $p + q > nq_0 + q_0$. *We have* $\mathfrak{g}$ *is decomposed under* $\mathfrak{l} + \mathfrak{u}(q_0) + \mathfrak{su}(p_1, q_1)$ *as in (10). There exists an element* $Z \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ *such that the positivity condition in Theorem 1.1 holds.*

4.2. **The homomorphism of** $\mathfrak{l} = \mathfrak{su}(n,1)$ **into** $\mathfrak{g} = \mathfrak{so}(2n,2), \mathfrak{so}^*(2n+2)$**.** These homomorphisms have been studied by Ihara [6] in terms of root systems. We shall need more precise formulation.

We shall realize $U(n,1)$ as symmetric subgroup of $G = SO^*(2n+2)$ and $SO(2n,2)$. The relevant decompositions of the Lie algebras $\mathfrak{g} = \mathfrak{so}^*(2n+2)$ and $\mathfrak{so}(2n,2)$ under $\mathfrak{u}(n,1)$ are parallell to the Cartan decomposition of the noncompact symmetric space $SO^*(2n+2)/U(n+1)$ and the compact symmetric space $SO(2n+2)/U(n+1)$. For completeness we provide details here.

We consider first the group $SO(2n,2)$. Let $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$ as a real space be equipped with the real indefinite form $\Re(x,y)_J = \Re(Jy)^* x$, where $(x,y)_J$ is the Hermitian form on $\mathbb{C}^{n+1}$ of signature $(n,1)$ in §2.1. The Lie group $G = SO(2n,2)$ is the connected component of the identity of the group $O(2n,2)$ defined by the real indefinite form. The symmetric space of $G = SO(2n,2)$ is the Lie ball, a complex structure being determined by fixing an element of the Lie algebra of $\mathfrak{so}(2)$. In particular $SU(n,1)$ is a subgroup of $SO(2n,2)$, and the corresponding embedding of the unit ball into the Lie ball is holomorphic by choosing a consistent complex structure on the ball. The multiplication by i, $z \to iz$ on $\mathbb{R}^{2n+2}$ induces an involution θ on the Lie algebra $\mathfrak{so}(2n,2), \theta^2 = id$. Thus $\mathfrak{so}(2n,2) = \ker(\theta - 1) \oplus \ker(\theta + 1)$ and it is clear that $\ker(\theta - 1) = \mathfrak{u}(n,1)$, the subspace of $\mathbb{C}$-linear transformations $T : z \mapsto Tz$ in $\mathfrak{so}(2n,2)$. The subspace $\ker(\theta+1)$ consists of anti-$\mathbb{C}$-linear transformations of the form $\alpha_A : z \mapsto A\bar{z}$ which are anti-symmetric with respect to the real form $\Re(z,w)_J$, i.e.,

$$\Re(A\bar{z}, w)_J = -\Re(z, A\bar{w})_J.$$

That is $\Re(Jw)^* A\bar{z} = -\Re(JA\bar{w})^* z = -\Re(JA\bar{w})^t \bar{z}$, namely $JA = -A^t J$. Putting $\tilde{A} = JA$ we have $\tilde{A}^t = -\tilde{A}$ and the action of an element $T \in \mathfrak{u}(n,1)$ on $\alpha_A \in \ker(\theta+1)$ is

$$[T, \alpha_A] : z \mapsto T\alpha_A z - \alpha_A T z = TA\bar{z} - A(\overline{Tz}) = (TA - A\bar{T})\bar{z}.$$

Hence $A \mapsto TA - A\bar{T}$. In terms of $\tilde{A}$ this is $\tilde{A} \mapsto J(TA - A\bar{T})$, which is

$$JTA - JA\bar{T} = -T^* JA - JA\bar{T} = -(T^*\tilde{A} + \tilde{A}\bar{T})$$

using $T \in \mathfrak{u}(n,1)$, $JT = -T^* J$. Namely $\mathfrak{u}(n,1)$ acts on $\ker(\theta+1)$ via the (real) representation on the space $V \wedge V$ of anti-symmetric $(n+1) \times (n+1)$ complex matrices.

We consider now the group $G = SO^*(2n+2)$. Let us fix its realization first. Let

$$j = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix},$$

where $J = J_{n,1} = \text{diag}(I_n, -1)$ is the matrix defining $U(n,1)$ in §2.1. The group G can be realized as

$$G = \{g \in GL(2n+2, \mathbb{C}); g^* j g = j, g^t g = I_{2n+2}, \det g = 1\}.$$

Note that G here is a different realization from the one in [3] but it will be convenient for our purpose. The group $SO^*(2n+2)$ is realized in [3, Ch. X, §2] as

$$G_1 = \{g \in GL(2n+2, \mathbb{C}); g^* j_1 g = j_1, g^t g = I_{2n+2}, \det g = 1\}.$$

where

$$j_1 = \begin{bmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{bmatrix}.$$

It is clear that j_1 and j are conjugate, $c^{-1} j c = j_1$, by an element c in $O(2n+2)$, which interchanges e_{n+1} and e_{2n+2}, so that the two groups G and G_1 are isomorphic by the map $g \in G_1 \mapsto cgc^{-1} \in G$.

Let $\tau : g \to jgj^{-1}$ be the involution on $GL(2n+2, \mathbb{C})$. Then τ maps G to G and is an involution on G and $\mathfrak{g}$. We claim that the set of fixed point is the group $U(n,1)$ and respectively $\mathfrak{u}(n,1)$. To be more precise let $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_- = \text{Ker}(\tau - 1) \oplus \text{Ker}(\tau + 1)$ be the decomposition of $\mathfrak{g}$ under τ. Then $\mathfrak{g}_+$ is a symmetric subalgebra of $\mathfrak{g}$ and by elementary matrix computations we find that the Lie algebra $\mathfrak{g}_+$ consists of real skew symmetric matrices

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \bar{X} = X, \quad X^t = -X,$$

satisfying

$$AJ = JD, \quad BJ = JB^t.$$

It is precisely the Lie algebra $\mathfrak{u}(n,1)$ under the identification of X with the complex matrix $A - iBJ$, i.e., $\mathfrak{g}_+ = \mathfrak{u}(n,1)$. The subspace $\mathfrak{g}_-$ consists of complex matrices of the form iX where X are real skew symmetric matrices

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \bar{X} = X, \quad X^t = -X,$$

with

$$AJ = -JD, \quad BJ = -JB^t.$$

We identify $\mathfrak{g}_-$ with the space $V \wedge V = \mathbb{C}^{n,1} \wedge \mathbb{C}^{n,1}$ of skew-symmetric matrices via the map $iX \mapsto A + iBJ$. The Lie algebra action of $\mathfrak{g}_+ = \mathfrak{u}(n,1)$ on $\mathfrak{g}_-$ is also the tensor product action of $\mathfrak{u}(n,1)$ on $V \wedge V$, by an elementary matrix computation of the Lie bracket in $\mathfrak{g}$. Summarizing we obtain the following

Lemma 4.2. *Under the above realizations of $SU(n,1)$ as a subgroup in $G = SO(2n,2)$ and $SO^*(2n+2)$ the Lie algebra $\mathfrak{g}$ has decomposition $\mathfrak{g} = \mathfrak{u}(n,1) \oplus (V \wedge V)_{\mathbb{R}}$ under $\mathfrak{l} = \mathfrak{su}(n,1)$. In particular the symmetric tensors $S^m V$ do not appear in the decomposition.*

4.3. **The Lie algebra $\mathfrak{g} = \mathfrak{sp}(n+1, \mathbb{R})$ and subalgebra $\mathfrak{l} = \mathfrak{su}(n, 1)$.** View $V = \mathbb{C}^{n+1}$ as a real space $V = \mathbb{R}^{2n+2}$ with the complex structure $v \to iv$ and the Hermitian form $(x, y)_J$ as in §2. Consider the following symplectic form

$$\omega(x, y) = \Im(Jx, y)$$

on V. We let $Sp(n+1, \mathbb{R}) =: Sp(n+1, \omega)$ be the symplectic group defined by ω. Clearly $U(n, 1)$ is a subgroup of $Sp(n+1, \mathbb{R})$ and the unit ball $U(n, 1)/U(n) \times U(1)$ is embedded holomorphically into the Siegel domain $Sp(n+1, \mathbb{R})/U(n+1)$. Symplectic embeddings of Hermitian symmetric spaces have been studied systematically by Satake [18].

The complex multiplication i on V defines an involution $A \mapsto i^{-1}Ai$ on all real linear transformations $A \in \mathrm{End}(\mathbb{R}^{2n+2})$, giving a decomposition of $\mathrm{End}(\mathbb{R}^{2n+2}) = M_{n+1,n+1}(\mathbb{C}) \oplus M_{n+1,n+1}(\mathbb{C})^c$ with $M_{n+1,n+1}(\mathbb{C})$ consisting of complex linear transformations $T : x \to Tx$ of $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$, and $M_{n+1,n+1}(\mathbb{C})^c$ of the conjugate complex linear transformations $T^c : x \to T\bar{x}$ where $T \in M_{n+1,n+1}(\mathbb{C})$. Restricting the involution to the Lie algebra $sp(n+1, \mathbb{R})$ we find

$$\mathfrak{sp}(n+1, \mathbb{R}) = \mathfrak{u}(n, 1) \oplus \mathfrak{r}$$

where

$$\mathfrak{r} = \{T^c; (JT)^t = JT\},$$

where A^t is the transpose of A. Indeed $\mathfrak{r}$ consists of complex conjugate transformations $T^c : x \to T\bar{x}$ in $\mathfrak{sp}(n+1, \mathbb{R})$, i.e., $\omega(T^c x, y) + \omega(x, T^c y) = 0$, $x, y \in V$; equivalently

$$\begin{aligned} 0 &= \Im(JT\bar{x}, y) + \Im(x, JT\bar{y}) = \Im y^* JT\bar{x} + \Im(JT\bar{y})^* x \\ &= \Im y^* JT\bar{x} + \Im y^t \overline{(JT)^t} x = \Im y^* JT\bar{x} - \Im y^* (JT)^t \bar{x}, \end{aligned}$$

implying $(JT)^t = JT$. As a representation space of $\mathfrak{u}(n, 1)$, $\mathfrak{r}$ is identified with the symmetric tensor power $S^2(V)$ of the standard representation V, concretely via $T^c \to JT \in S^2(V)$. We have further, via this identification,

$$\mathfrak{sp}(n+1, \mathbb{R}) = \mathfrak{su}(n, 1) \oplus \mathbb{R}Z_0 \oplus \mathfrak{r} = \mathfrak{su}(n, 1) \oplus \mathbb{R}Z_0 \oplus (S^2(V))_{\mathbb{R}}$$

where the element Z_0 generates the centralizer of $\mathfrak{su}(n, 1)$ in $\mathfrak{u}(n, 1)$ defined as the multiplication by i on $V_{\mathbb{R}}$. The Lie bracket on $\mathfrak{r}$ is given, by the definition,

$$[X^c, Y^c] = U, \quad U = X\bar{Y} - Y\bar{X} \in \mathfrak{u}(n, 1).$$

In particular we have

$$[X^c, iX^c] = -2iX\bar{X}.$$

The Killing form on $\mathfrak{sp}(n+1, \mathbb{R})$ is $B(X, Y) = \mathrm{tr}_{\mathbb{R}} XY$ by our normalization. Now if $X, Y \in S^2(V)$ is in the last component $S^2(V_1) \subset S^2(V)$ in (7), then $(JX)^t = X^t = JX = X$ since $J = I$ on V_1. Hence we have

$$B(Z_0, [X^c, iX^c]) = \mathrm{tr}_{\mathbb{R}} i([X^c, iX^c]) = \mathrm{tr}_{\mathbb{R}} i(-2iX\bar{X}) = 2\mathrm{tr}_{\mathbb{R}} X\bar{X} = 2\mathrm{tr}_{\mathbb{R}} XX^* = 4\mathrm{tr}_{\mathbb{C}} XX^*,$$

and is positive definite. Summarizing we have

Lemma 4.3. *The Lie algebra $\mathfrak{sp}(n+1,\mathbb{R})$ has a decomposition $\mathfrak{g} = \mathfrak{sp}(n+1,\mathbb{R}) = \mathfrak{su}(n,1) \oplus \mathbb{R}Z_0 \oplus (S^2(V))_{\mathbb{R}} = \mathfrak{l} \oplus \mathbb{R}Z_0 \oplus (S^2(V))_{\mathbb{R}}$ under $\mathfrak{l}$. The $\mathfrak{l}$-invariant bilinear skew-symmetric form $(X^c, Y^c) \to B(Z_0, [X^c, Y^c])$ on $S^2(V)$ induces a positive definite quadratic form $X^c \to B(Z_0, [X^c, iX^c])$ on the subspace $S^2(V_1) \subset S^2(V)$.*

Remark 4.4. *Consider a natural homomorphism of $L = SU(2,1)$ into the exceptional Lie group $G = F_{4(-20)}$ inducing a totally geodesic embedding of the complex unit ball L/M in $\mathbb{C}^2$ into G/K realized as the unit ball in the octonian $\mathbb{O}^2$ (see e.g.* [7, Theorem 2.2]*). We can show in this case that $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{m}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{l}) = \mathfrak{su}(3)$. But we observe that for any element $Z \in \mathfrak{z}_{\mathfrak{k}}(\mathfrak{m}) = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{l})$ the positivity condition (4) is never fulfilled. It is thus an open question to know whether the local rigidity holds in this case.*

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